Pólya permanent problem: 100 years after

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Joint work with

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1. Gregor Dolinar, Alexander E. Guterman, Bojan Kuzma, Marko Orel, On the Polya permanent problem over finite fields, European J. of Combinatorics, 32, 2011, 116-132

2. Gregor Dolinar, Alexander E. Guterman, Bojan Kuzma, On Gibson barrier for Polya problem, Fundamental and Applied Mathematics, 16(8), 2010, 73-86

3. Gregor Dolinar, Alexander E. Guterman, Bojan Kuzma, Pólya convertibility problem for symmetric matrices, Math. Notes, 92 (5), 2012, 684-698.

4. Mikhail V. Budrevich, Alexander E. Guterman, Permanent has less zeros than determinant over finite fields, American Mathematical Society, Contemporary Mathematics, 579, 2012, 33-42.

5. Mikhail V. Budrevich, Alexander E. Guterman, On the Gibson bounds over finite fields, Serdica Math. J. 38, 2012, 395416

6. Gregor Dolinar, Alexander E. Guterman, Bojan Kuzma, Marko Orel, Permanent versus determinant over a finite field, Journal of Mathematical Sciences (New-York), 193(3), 2013, 404-412

$$\det A = \sum_{\sigma \in \mathfrak{S}_n} sgn(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

and

$$\operatorname{per} A = \sum_{\sigma \in \mathfrak{S}_n} a_{1\sigma(1)} \cdots a_{n\sigma(n)},$$

here $A = (a_{ij}) \in M_n(\mathbb{C})$, $\mathfrak{S}_n$ denotes the set of all permutations of the set $\{1, 2, \ldots, n\}$. The value $sgn(\sigma) \in \{-1, 1\}$ is the signum of the permutation σ.

per is a combinatorial invariant:

$$\operatorname{per}(PAQ) = \operatorname{per} A$$

for all permutation matrices P, Q

Some applications of permanent

Derangements problem

In how many ways can a dance be arranged for n married couples, so that no husband dances with his own wife?

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$$D_n = \operatorname{per}\begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & 0 \end{pmatrix} = \operatorname{per}(J_n - I_n) = n! \cdot \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

Ménage problem or problème des ménages

In how many ways can n married couples be placed at a round table, so that men and women sit in alternate places and no husband sit on either side of his wife?

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$$U_n = \text{per}\begin{pmatrix} 0 & 0 & 1 & \dots & 1 & 1 \\ 1 & 0 & 0 & \ddots & \vdots & \vdots \\ 1 & 1 & 0 & \ddots & 1 & 1 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 1 \\ 1 & 1 & \dots & 1 & 0 & 0 \\ 0 & 1 & 1 & \dots & 1 & 0 \end{pmatrix} = \text{per}(J_n - I_n - P_n)$$

P_n is a permutation matrix of $(1,2)(2,3)\cdots(n-1,n)(n,1)$.

Ménage problem or problème des ménages

In how many ways can n married couples be placed at a round table, so that men and women sit in alternate places and no husband sit on either side of his wife?

Sequence number $A059375$ in on-line encyclopedia of integer sequences

The first terms:

$12, 96, 3120, 115200, 5836320, 382072320, 31488549120, \ldots$

Ménage problem or problème des ménages

Formulated in 1891 by Édouard Lucas and independently, a few years earlier, by Peter Guthrie Tait in connection with knot theory

Touchard (1934) derived the formula

$$U_n = 2 \cdot n! \sum_{k=0}^{n} (-1)^k \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)!$$

Latin squares

S is a set, $|S| = n$ usually, $S = \{1, 2, \ldots, n\}$

A Latin rectangle on S is an $r \times s$ matrix A with $a_{ij} \in S$, $a_{ij} \neq a_{il}$, and $a_{ij} \neq a_{kj}$.

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$n \times n$ Latin rectangle is a Latin square.

Problems: 1. To find the number $L(n, n)$ of Latin squares on S

2. To find the number $L(r, n)$ of $r \times n$ Latin rectangles on S

Known facts

1. $L(1, n) = 1$

2. $L(2, n) = n! \cdot D_n$

3. $L(3, n) = n! \cdot \sum_{k=0}^{\lfloor n/2 \rfloor} C_n^k D_{n-k} D_k U_{n-2k}$

Λ_n^k is the set of (0,1)-matrices with k 1 in each row and column.

$m(k, n)$ and $M(k, n)$ are lower and upper bounds for permanent in Λ_n^k.

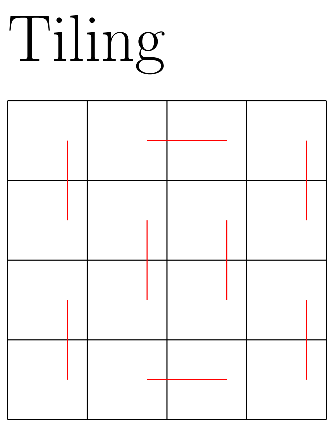
Then

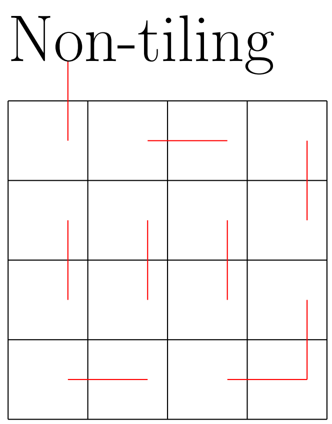
$$n! D_n \prod_{t=2}^{r-1} m(n-t, n) \le L(r, n) \le n! D_n \prod_{t=2}^{r-1} M(n-t, n)$$

Domino tiling

Consider $m \times n$ rectangular chessboard and 2×1 dominoes.

A tiling is a placement of dominoes that covers all the cells of the board perfectly.





1. If there exists a tiling if we consider a usual chess-board with one corner-cell deleted?

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NO. The total number of cells is odd.

2. If there exists a tiling if we consider a usual chess-board with two opposite corner-cells deleted?

NO. Both deleted cells are of the same color, but domino covers two cells of different colors

Problems:

1. *Existence* of tilings.

2. If there are tilings, how many are them?

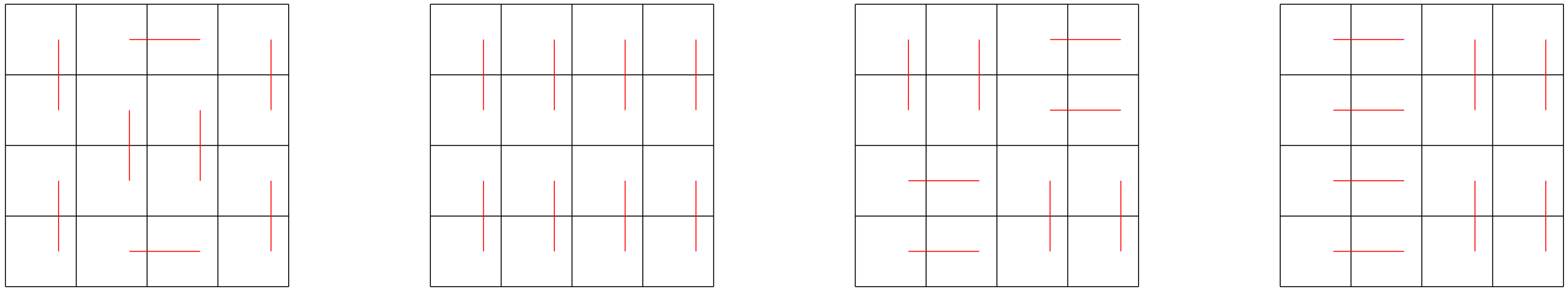
Problems:

1. For which m, n do there $\exists$ tilings?

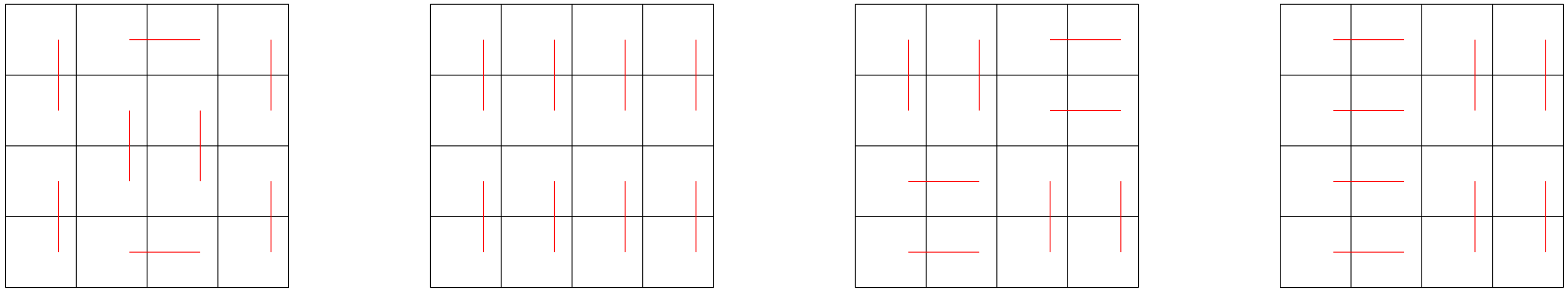
2. If there are tilings, how many are them?

Theorem. Tiling exist $\Leftrightarrow$ m, n are NOT both odd (i.e. mn is even).

Example.



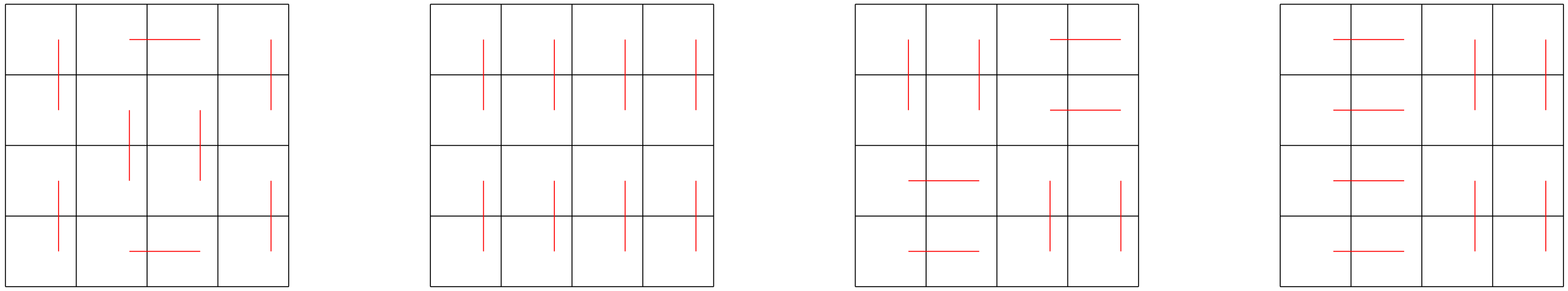
Example.



$$T(2,n) = T(2,n-1) + T(2,n-2)$$

$$T(3,2n) = 4T(3,2n-2) - T(3,2n-4)$$

Example.



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Difficult recurrent formulas...

Perfect matching in a graph is a selection of edges that covers each vertex exactly once.

tilings $\longleftrightarrow$ perfect matchings in underlying grid graph

Chessboard coloring $\Longrightarrow$ bipartite graph

Bipartite graph $\Longrightarrow$ adjacency matrix A

The number of tilings $=$ number of perfect matchings $= \mathrm{per}(A)$

The number of tilings: Temperley & Fisher (1961) and Kasteleyn (1961)

$$\prod_{j=1}^{m}\prod_{k=1}^{n}\left(4\cos^2\frac{\pi j}{m+1}+4\cos^2\frac{\pi k}{n+1}\right)^{\frac{1}{4}}$$

equivalent to

$$\prod_{j=1}^{\lceil\frac{m}{2}\rceil}\prod_{k=1}^{\lceil\frac{n}{2}\rceil}\left(4\cos^2\frac{\pi j}{m+1}+4\cos^2\frac{\pi k}{n+1}\right).$$

If m or n is 2: the sequence reduces to the Fibonacci sequence (sequence A000045 in OEIS) (Klarner & Pollack 1980)

Applications of permanent:

Counting function for combinatorial problems

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Makes everybody happy

	det	per
Geometry	Oriented volume	Combinatorial geometry
Algebra	$\lambda_1 \cdots \lambda_n$	Bounds
Complexity	$O(n^3)$	$\sim (n-1) \cdot (2^n - 1)$

Ryser's formula

$$\operatorname{per}(A) = \sum_{t=0}^{n-1} (-1)^t \sum_{X \in \Lambda_{n-t}} \prod_{i=1}^{n} r_i(X)$$

$r_i(X) = \sum_{j=1}^{t} x_{ij}$ — ith row sum

Λ_{n-t} — the set of all $n \times (n-t)$ submatrices of A

How many tilings ?

To compute permanent is HARD!

Even if the entries are just $0, 1$, computing the permanent is $\sharp P$-complete.

The quantity of transformations preserving a given matrix invariant provide a "measure" of its complexity

Theorem 1 [Frobenius, 1896]

$$T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$$

— linear, bijective

$$\det(T(A)) = \det A \qquad \forall A \in M_n(\mathbb{C})$$

$$\Downarrow$$

$$\exists P, Q \in GL_n(\mathbb{C}), \det(PQ) = 1 :$$

$$T(A) = PAQ \quad \forall A \in M_n(\mathbb{C}) \text{ or } T(A) = PA^tQ \quad \forall A \in M_n(\mathbb{C})$$

Theorem 2 [Marcus, May] Linear transformation T is permanent preserver iff

$$T(A) = P_1 D_1 A D_2 P_2 \quad \forall A \in M_n(\mathbb{F}), \text{ or}$$

$$T(A) = P_1 D_1 A^t D_2 P_2 \quad \forall A \in M_n(\mathbb{F})$$

here D_i are invertible diagonal matrices, $i = 1, 2$

P_i are permutation matrices, $i = 1, 2$

Polya, 1913 observed:

$n = 2$:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \operatorname{per} \begin{pmatrix} a & b \\ -c & d \end{pmatrix}$$

Problem 1. Polya, 1913. Does $\exists$ a uniform way of affixing $\pm$ to the entries of $A = (a_{ij}) \in M_n(\mathbb{F})$: $\mathrm{per}(a_{ij}) = \det(\pm a_{ij})$?

$n = 2$: $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ -c & d \end{pmatrix}$$

Szegö, 1914. $n > 2$: NO.

Why NOT ?

$n = 3$: consider $J_3 = \left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix}\right)$.

Then $\operatorname{per} J_3 = 6$ but

$$\det \begin{pmatrix} \pm 1 & \pm 1 & \pm 1 \\ \\ \pm 1 & \pm 1 & \pm 1 \\ \\ \pm 1 & \pm 1 & \pm 1 \end{pmatrix} < 6$$

since each -1 is in two summands, so all 6 summands can not be positive.

What about SUBSETS of M_n?

Sometimes the conversion is possible:

1. $$\begin{pmatrix} a & b & 0 \\ c & d & e \\ f & g & h \end{pmatrix} \mapsto \begin{pmatrix} a & b & 0 \\ -c & d & e \\ f & -g & h \end{pmatrix}$$

2. A: $a_{ij} = 0$ if $j - i \geq 2$ (Hessenberg matrices)

$$A \mapsto \tilde{A} = (\tilde{a_{ij}}):\ \tilde{a_{ij}} = \begin{cases} -a_{ij}, & \text{if } j - i = 1 \\ a_{ij}, & \text{otherwise} \end{cases}$$

3. A is Jacobi (3-diagonal) matrix.

$A \mapsto \widehat{A} = (\widehat{a_{ij}})$:

$$\widehat{a_{st}} = \begin{cases} \mathrm{i}\, a_{st}, & \text{if } s \neq t \\ a_{ss}, & \text{if s=t} \end{cases}$$

Problem 2. Under what conditions does there exist a transformation

$\Phi : M_n(\mathbb{F}) \to M_m(\mathbb{F})$ satisfying

$$\operatorname{per} A = \det \Phi(A)?$$

Here a transformation Φ on $M_n(\mathbb{F})$ is called a converter.

Are there linear transformations of this type ?

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Theorem (Marcus, Minc, 1961). *There is no bijective linear transformation* $\Phi : M_n(\mathbb{F}) \to M_n(\mathbb{F}), n > 2$ *satisfying* $\operatorname{per} A = \det \Phi(A)\ \forall A \in M_n(\mathbb{F})$.

Proof: based on linear algebra.

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Proof: based on linear algebra.

Theorem (J. von zur Gathen, 1987). *Let* $\mathbb{F}$ *be infinite,* $char(\mathbb{F}) \neq 2$. *There is no bijective affine transformation* $\Phi : M_n(\mathbb{F}) \to M_n(\mathbb{F}), n > 2$ *satisfying* $\operatorname{per} A = \det \Phi(A)$ $\forall$ $A \in M_n(\mathbb{F})$.

Proof: based on algebraic geomery.

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Example. There are non-bijective non-linear converters $\Phi : M_n(\mathbb{F}) \to M_m(\mathbb{F})$ of *per* and *det*:

$$\Phi : A \mapsto \begin{pmatrix} 1 & \frac{1}{2}(\det A - \operatorname{per} A) \\ \\ 1 & \frac{1}{2}(\det A + \operatorname{per} A) \end{pmatrix} \oplus \mathrm{Id}_{m-2}\,.$$

Hence, $\operatorname{per} A = \det \Phi(A)$ and $\det A = \operatorname{per} \Phi(A)$.

Example. There are bijective non-linear converters of *per* and *det* over infinite fields:

For any $\mathbb{F}$ and any $\lambda, \mu \in \mathbb{F}$

$$\operatorname{card}\{A \in M_n(\mathbb{F}) | \det A = \mu,\ \operatorname{per} A = \lambda\} =$$

$$= \operatorname{card} \mathbb{F}$$

$$= \operatorname{card}\{A \in M_n(\mathbb{F}) | \det A = \lambda,\ \operatorname{per} A = \mu\},$$

thus partial bijections exist, and hence the bijection exists.

WHAT HAPPENS OVER FINITE FIELDS ?

Theorem. [Dolinar, Guterman, Kuzma, Orel] **For any $n \geq 3$ there exists $q_0 = q_0(n)$ such that for any finite field $\mathbb{F}$, $\operatorname{ch}\mathbb{F} \neq 2$, $|\mathbb{F}| \geq q_0$ there are NO bijective maps $\Phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ satisfying**

$$\operatorname{per} A = \det \Phi(A). \tag{1}$$

If $n = 3$ the conclusion holds for any finite field with $\operatorname{ch}\mathbb{F} \neq 2$.

$$|D_n| = |M_n| - |GL_n|$$

$$\Downarrow$$

if $n \geq 4$

$$|D_n| = q^{n^2} - \prod_{k=1}^{n}(q^n - q^{k-1}) = q^{n^2-1} + q^{n^2-2} + O(q^{n^2-5})$$

$$|D_n| = q^{n^2} - \prod_{k=1}^{n}(q^n - q^{k-1}) = q^{n^2-1} + q^{n^2-2} + 0 + 0 + O(q^{n^2-5})$$

$$L_n = q^{n^2-1} - q^{n^2-2} + O(q^{n^2-3}) \qquad (n \geq 4),$$

$$U_n = q^{n^2-1} + 0 + O(q^{n^2-3}) \qquad (n \geq 4).$$

$$L_n \leq P_n \leq U_n < D_n$$

$|P_n| \leq U_n < |D_n|$ if q is sufficiently large ($q \geq q_0$).

$$U_n = q^{n^2-1} + O(q^{n^2-3})$$

$$|D_n| = q^{n^2-1} + q^{n^2-2} + O(q^{n^2-5})$$

n	3	4	5	6	7	8	9	10	11
q_0	3	43	79	121	167	223	289	367	449
n	12	13	14	15	16	17	18	19	20
q_0	541	641	751	877	997	1151	1279	1433	1597

Probability

[P. Erdös, A. Rényi] What is the probability of the permanent of a given matrix to be equal to 0?

Theorem. *Let* $\mathbb{F}$ *be a finite field,* $\operatorname{ch}\mathbb{F} \neq 2$*.* $\forall\ \lambda \in \mathbb{F}$

$$P(\operatorname{per} A = \lambda) = \frac{1}{q} + O(\frac{1}{q^2}).$$

Let us consider tensor of permanent of $A \in M_{k,n}$, $k \leq n$ which is defined by

$$T_A^{i_1,\ldots,i_{n-k}} = \begin{cases} \operatorname{per}(A(|i_1,\ldots,i_{n-k})), & \text{if all } i_1,\ldots,i_{n-k} \text{ are different} \\ 0, & \text{otherwise.} \end{cases}$$

Examples:

1. $k = n$. Then $T_A = \operatorname{per} A$.

2. $k = 1$, $A = (a_1, \ldots, a_n)$. Then $T_A^{1,\ldots,i-1,i+1,\ldots,n} = a_i$.

Properties:

1. $A \in M_{1,n}$ is a vector. Then $T_A \equiv 0$ if and only if $A \equiv 0$.

2. For any A it holds T_A is symmetric.

Definition. The convolution of T_B, $B \in M_{k,n}$ and $x \in \mathbb{F}_q^n$ is

$$(T_B \circ x)^{i_1,\dots,i_{n-k-1}} = \sum_{j=1}^{n} T^{i_1,\dots,i_{n-k-1},j} \cdot x_j \text{ of the valency } (n-k-1).$$

Lemma. Let $a \in \mathbb{F}_q^n$, $A \in M_{k,n}, k < n$, $B = \left(\begin{smallmatrix} a \\ A \end{smallmatrix}\right)$. Then $T_B = T_A \circ a$.

Corollary. For $A \in M_n(\mathbb{F}_q)$ formed by the rows $a_1, \dots, a_n$.

$$\mathrm{per}(A) = T_A = T_{\begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{pmatrix}} = T_{\begin{pmatrix} a_2 \\ a_3 \\ \dots \\ a_n \end{pmatrix}} \circ a_1 = \left(T_{\begin{pmatrix} a_3 \\ \dots \\ a_n \end{pmatrix}} \circ a_2\right) \circ a_1 =$$

$$= \dots = (\dots(T_{a_n} \circ a_{n-1}) \circ a_{n-2} \dots) \circ a_1$$

Lemma. Let $A \in M_{k\times n}(\mathbb{F}_q)$ and $T_A \not\equiv 0$. Then there are at least $q^n - q^k$ different vectors $x \in F_q^n$ such that $R = T_A \circ x \not\equiv 0$.

Lemma. Let $a = (1, \ldots, 1) \in \mathbb{F}_q^n$, $n \geq 3$. Then the number of vectors $x \in \mathbb{F}_q^n$ such that $R = T_a \circ x \neq 0$ is equal to $q^n - 1 > q^n - q$.

Theorem (Budrevich, Guterman). *Let* $\mathbb{F}$ *be a finite field,* $\mathrm{ch}\,\mathbb{F} \neq 2$.

$\forall \ \ n \geq 3$ *the number of zeros of* per *is less*

than the number of zeros of $\det$.

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Theorem (Budrevich, Guterman). *Let* $\mathbb{F}$ *be a finite field,* $\operatorname{ch}\mathbb{F} \neq 2$.

$\forall$ $n \geq 3$ *there is NO bijective map* $T : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ *satisfying*

$$\operatorname{per} A = \det T(A).$$

Problem 3 (Polya)**.** Given a (0,1)-matrix $A \in M_n(\mathbb{F})$, does $\exists$ B, obtained by changing some of the $+1$ entries of A into -1, so that

$$\operatorname{per} A = \det B?$$

The following problems are equivalent to the problem above:

1. Even cycle: A digraph. Does it have no directed circuits of even length?

2. Sign solvability: When does a real square matrix have the property that every real matrix with the same sign pattern is non-singular?

. . . .

There are more than 30 equivalent problems of this kind, see [W. McCuaig, *The Electronic Journal of Combinatorics* **11** (2004), R79].

Let M_n be the set of all $n \times n$ $\{0, 1\}$ matrices over $\mathcal{R}$ — any ring of characteristic 0.

$\mathrm{S}_n \subseteq M_n$ — subset of symmetric matrices.

$v(A)$ is the number of 1 of A. NB: $v(A) = \sum$ all entries of A.

$X \circ A$ denote the Schur (entrywise) product of two matrices.

Definition.

$A \in M_n$ is *convertible* if $\exists\ X \in M_n(\pm 1)$:

$$\operatorname{per} A = \det(X \circ A) \qquad .$$

$A \in \mathrm{S}_n$ is *symmetrically convertible*, if $\exists\ X \in \mathrm{S}_n(\pm 1)$:

$$\operatorname{per} A = \det(X \circ A) \qquad .$$

$A \in \mathrm{S}_n$ is *symmetrically weakly-convertible*, if $\exists\ X \in \mathrm{S}_n(\pm 1)$:

$$\operatorname{per} A = |\det(X \circ A)| \qquad .$$

OBSERVATION

$A \in \mathrm{S}_n$ is *symmetrically weakly-convertible*.

Then A is *convertible*.

Multiply a row of A with -1.

Can a matrix with arbitrary number of units be convertible ?

Theorem. [Gibson, 1971] *Let* $A \in M_n$ *be a convertible matrix with* $\operatorname{per} A > 0$.

Then $v(A) \leq \Omega_n := \dfrac{n^2 + 3n - 2}{2}$.

The equality holds $\Leftrightarrow$ $\exists$ *permutation matrices* P, Q*:* $A = PT_nQ$.

Here *Gibson matrix* $T_n = (t_{ij}) \in M_n$ is $t_{ij} = \begin{cases} 0, & \text{if } 1 \le i < j < n \\ 1, & \text{if } i \ge j \text{ or } j = n \end{cases}.$

Also $G_n = (g_{ij}):\quad g_{ij} = \begin{cases} 0, & \text{if } i + j \le n - 1 \\ 1, & \text{if } i + j > n - 1 \end{cases}$

Note that $G_n = T_n Q_n$ for $Q_n = Q(\sigma)$ s.t.

$\sigma = (1, n-1)(2, n-2), \ldots, (\lfloor n/2 \rfloor, \lfloor (n+1)/2 \rfloor),$

here $\lfloor x \rfloor$ is the largest integer $\le x$.

$$T_5 = \begin{pmatrix} 1&0&0&0&1 \\ 1&1&0&0&1 \\ 1&1&1&0&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \end{pmatrix} \qquad G_5 = \begin{pmatrix} 0&0&0&1&1 \\ 0&0&1&1&1 \\ 0&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \end{pmatrix}$$

The following results are from [Dolinar, Guterman, Kuzma]

Theorem. $n \geq 3$, $A \in \mathrm{S}_n$, $\operatorname{per} A > 0$, A *is convertible*.

Then $v(A) \leq \Omega_n = \dfrac{n^2 + 3n - 2}{2}$.

Let $v(A) = \Omega_n$ *then* A *is convertible* $\Leftrightarrow$ $A = PG_nP^t$ *for some permutation matrix* P.

Symmetric convertibility of matrices with maximal number of units ?

Theorem (Dolinar, Guterman, Kuzma). $n \geq 3$, $A \in S_n$, $\operatorname{per} A > 0$, $v(A) = \Omega_n = \frac{n^2+3n-2}{2}$ *and* A *is convertible. Then*

$n \neq 2 \pmod 4 \Longrightarrow A$ *is symmetrically convertible.*

$n = 2 \pmod 4 \Longrightarrow A$ *is symmetrically weakly-convertible, but not symmetrically convertible.*

Can we find ω_n s.t. $\forall$ A: $v(A) < \omega_n \Rightarrow A$ is convertible ?

$\omega_n = n + 5$

Theorem. *[Little, 1972, Graph Theory approach]* $n \geq 2$, $A \in M_n$, $v(A) \leq n + 5 \Rightarrow A$ *is convertible.*

Is $n+5$ a really magic number ?

Theorem (Dolinar, Guterman, Kuzma). $n \geq 3$, $A \in M_n$, $v(A) = n+6$.

Then A *is not convertible* $\Leftrightarrow$ $\exists$ *permutation matrices* P, Q*:* $PAQ = \mathrm{Id}_{n-3} \oplus J_3$*, where* $J_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

What is about S_n ?

Theorem. [Dolinar, Guterman, Kuzma]

1. $n \geq 2$, $A \in \mathrm{S}_n$, $v(A) \leq n+5 \Rightarrow A$ *is symmetrically weakly-convertible.*

2. $n \geq 3$, $v(A) = n+6$. *Then* A *is not convertible* $\Leftrightarrow \exists$ *permutation matrices* P, Q*:* $PAQ = \mathrm{Id}_{n-3} \oplus J_3$.

3. A *is convertible,* $v(A) = n+6$, $\Rightarrow A$ *is symmetrically weakly-convertible.*

What happens in between ω_n and Ω_n ?

Theorem. [Dolinar, Guterman, Kuzma] *Let* $r \in \mathbb{Z}$*:* $\omega_n \leq r \leq \Omega_n$.

Then

1. $\exists$ $A \in \mathrm{S}_n$*: symmetrically weakly-convertible*, $\mathrm{per}(A) \neq 0$, $v(A) = r$

2. $\exists$ $B \in \mathrm{S}_n$*: not convertible*, $v(B) = r$

For fully indecomposable matrices lower bound can be improved.

For fully indecomposable matrices lower bound can be improved.

$A \in M_n$ is decomposable, if $\exists$ a permutation matrix $P \in M_n$ such that

$$A = P \begin{pmatrix} B & 0 \\ C & D \end{pmatrix} P^t, \text{ where } B, D \text{ are square.}$$

If A is not decomposable, it is called indecomposable.

$A \in M_n$ is partially decomposable if $\exists$ permutation matrices $P, Q \in M_n$

such that

$$A = P \begin{pmatrix} B & 0 \\ C & D \end{pmatrix} Q,$$

where B, D are square.

If A is no partially decomposable, it is fully indecomposable.

Note, O is decomposable and partially decomposable.

Lemma.

- If $A \in M_n$ is decomposable, then A is partially decomposable.
- If $A \in M_n$ is fully indecomposable, then A is indecomposable.

Example.

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in M_2$$

is indecomposable, but partially decomposable with $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Q = I$.

For fully indecomposable matrices lower bound can be improved.

Theorem (Budrevich, Dolinar, Guterman, Kuzma). *Let* $A \in M_n$ *be fully indecomposable,* $v(A) \leq 2n+2$*. Then* A *is convertible.*

Example. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Then A is fully indecomposable, non-convertible, and $v(A) = 9 = 2 \cdot 3 + 3$.

Example. Let

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & & & & \ddots & \ddots & & \vdots \\ 0 & 0 & \ldots & \ldots & \ldots & 0 & 1 & 0 \\ 0 & 0 & \ldots & \ldots & \ldots & \ldots & 0 & 1 \\ 1 & 0 & \ldots & \ldots & \ldots & \ldots & 0 & 0 \end{pmatrix}$$

Then A is not fully indecomposable, indecomposable, not convertible, and $v(A) = n + 6$.

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Important trivial observation:

Zeros are better than ones since they are stable under the sign operation!