On the BLUEs in Two Linear Models via C. R. Rao's Pandora's Box

Nesrin Güler [1],

Simo Puntanen [2] & **Halim Özdemir** [3]

[1]Department of Statistics, Sakarya University, Sakarya, Turkey
[2]School of Information Sciences, University of Tampere, Finland
[3]Department of Mathematics, Sakarya University, Sakarya, Turkey

23rd **International Workshop on Matrices and Statistics**

June 08–12, 2014

Ljubljana, SLOVENIA

Outline

1. Introduction

Consider the partitioned linear model

$$\mathbf{y} = \mathbf{X}\beta + \varepsilon = \mathbf{X}_1\beta_1 + \mathbf{X}_2\beta_2 + \varepsilon,$$

denoted as

$$\mathcal{M} = \{\mathbf{y}, \mathbf{X}\beta, \mathbf{V}\} = \{\mathbf{y}, \mathbf{X}_1\beta_1 + \mathbf{X}_2\beta_2, \mathbf{V}\}, \tag{1.1}$$

- $\mathbf{y} \in \mathbb{R}^{n\times 1}$ is an observable random vector with expectation $\mathrm{E}(\mathbf{y}) = \mathbf{X}\beta$ and covariance matrix $\mathrm{cov}(\mathbf{y}) = \mathbf{V}$,
- $\mathbf{X} \in \mathbb{R}^{n\times p}$ is a known matrix partitioned columnwise as $\mathbf{X} = (\mathbf{X}_1 : \mathbf{X}_2)$ with $\mathbf{X}_i \in \mathbb{R}^{n\times p_i}$, $i = 1, 2$,
- $\beta \in \mathbb{R}^{p\times 1}$ is a vector of fixed unknown parameters with $\beta = (\beta_1' : \beta_2')'$ and $p = p_1 + p_2$,
- $\varepsilon \in \mathbb{R}^{n\times 1}$ is a vector of random errors,
- $\mathbf{V} \in \mathbb{R}^{n\times n}$ is a known nonnegative definite matrix (nnd).

Estimability

$\mathbf{K}\beta$ *is estimable under the model* $\mathcal{M}$ *if and only if*

$$\mathscr{C}(\mathbf{K}') \subseteq \mathscr{C}(\mathbf{X}').$$

$\mathbf{K}_1\beta_1$ *is estimable under* $\mathcal{M}$ *if and only if*

$$\mathbf{K}_1 = \mathbf{L}\mathbf{M}_2\mathbf{X}_1$$

for some matrix $\mathbf{L}$.

Alalouf & Styan (1979), Gross & Puntanen (2000, Lemma 1) and Isotalo & Puntanen (2009).

$\mathbf{X}_1\beta_1$ *is estimable under* $\mathcal{M}$ *if and only if*

$$\mathscr{C}(\mathbf{X}_1) \cap \mathscr{C}(\mathbf{X}_2) = \{\mathbf{0}\}.$$

Puntanen, Styan & Isotalo (2011, p.160, 345) and Tian & Zhang (2011).

The best linear unbiased estimator (BLUE)

The BLUE *of* $\mathbf{X}\beta$ *under* $\mathcal{M}$*, denoted as* $\text{BLUE}(\mathbf{X}\beta \mid \mathcal{M})$*, is defined to be an unbiased linear estimator* $\mathbf{Gy}$ *such that its covariance matrix* $\text{cov}(\mathbf{Gy})$ *is minimal, in the Löwner sense, among all covariance matrices* $\text{cov}(\mathbf{Fy})$ *such that* $\mathbf{Fy}$ *is unbiased for* $\mathbf{X}\beta$*.*

$$\mathbf{Gy} = \text{BLUE}(\mathbf{X}\beta \mid \mathcal{M}) \iff \mathbf{G}(\mathbf{X} : \mathbf{VM}) = (\mathbf{X} : \mathbf{0}), \tag{1.2}$$

where $\mathbf{M} = \mathbf{I}_n - \mathbf{P}_{\mathbf{X}}$, see, e.g., Rao (1967), Zyskind (1967), Baksalary & Trenkler (2009, 2011).

The observed value of $\mathbf{Gy}$ is unique wp 1 if and only if the model $\mathcal{M}$ is consistent, i.e.,

$$\mathbf{y} \in \mathscr{C}(\mathbf{X} : \mathbf{V}) = \mathscr{C}(\mathbf{X} : \mathbf{VM}) \tag{1.3}$$

holds wp 1; see, e.g., Rao (1973b, p.282), Puntanen & Styan (1990), and Baksalary, Rao & Markiewicz (1992). The corresponding consistency is assumed in all models considered in this study.

$$\mathbf{Ay} = \mathrm{BLUE}(\mathbf{K}\boldsymbol{\beta} \mid \mathcal{M}) \iff \mathbf{A}(\mathbf{X} : \mathbf{VM}) = (\mathbf{K} : \mathbf{0}).$$

Moreover, if $\mathbf{X}_1\boldsymbol{\beta}_1 = (\mathbf{X}_1 : \mathbf{0})\boldsymbol{\beta}$ is estimable under the model $\mathcal{M}$, then

$$\mathbf{G}_1\mathbf{y} = \mathrm{BLUE}(\mathbf{X}_1\boldsymbol{\beta}_1 \mid \mathcal{M}) \iff \mathbf{G}_1(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{VM}) = (\mathbf{X}_1 : \mathbf{0} : \mathbf{0}). \quad (1.4)$$

$$\mathbf{Gy} = \mathrm{BLUE}(\mathbf{X}\beta \mid \mathcal{M}) \iff \mathbf{G}(\mathbf{X} : \mathbf{VM}) = (\mathbf{X} : \mathbf{0})$$

Lemma (1.1)

Consider the linear model $\mathcal{M} = \{\mathbf{y}, \mathbf{X}\beta, \mathbf{V}\}$*. Then* $\mathbf{Gy}$ *is the* BLUE *for* $\mathbf{X}\beta$ *under* $\mathcal{M}$ *if and only if there exists a matrix* $\mathbf{L} \in \mathbb{R}^{p \times n}$ *such that* $\mathbf{G}$ *is solution to*

$$\mathbf{\Gamma}_{\mathcal{M}} \begin{pmatrix} \mathbf{G}' \\ \mathbf{L} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{X}' \end{pmatrix}, \tag{1.5}$$

where $\mathbf{\Gamma}_{\mathcal{M}} = \left(\begin{smallmatrix} \mathbf{V} & \mathbf{X} \\ \mathbf{X}' & \mathbf{0} \end{smallmatrix}\right)$ *(Rao, 1971).*

$$\mathbf{Gy} = \mathrm{BLUE}(\mathbf{X}\beta \mid \mathcal{M}) \Longleftrightarrow \mathbf{G}(\mathbf{X} : \mathbf{VM}) = (\mathbf{X} : \mathbf{0})$$

Lemma (1.1)

Consider the linear model $\mathcal{M} = \{\mathbf{y}, \mathbf{X}\beta, \mathbf{V}\}$. *Then* $\mathbf{Gy}$ *is the* BLUE *for* $\mathbf{X}\beta$ *under* $\mathcal{M}$ *if and only if there exists a matrix* $\mathbf{L} \in \mathbb{R}^{p\times n}$ *such that* $\mathbf{G}$ *is solution to*

$$\mathbf{\Gamma}_{\mathcal{M}} \begin{pmatrix} \mathbf{G}' \\ \mathbf{L} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{X}' \end{pmatrix}, \tag{1.5}$$

where $\mathbf{\Gamma}_{\mathcal{M}} = \left(\begin{smallmatrix} \mathbf{V} & \mathbf{X} \\ \mathbf{X}' & \mathbf{0} \end{smallmatrix}\right)$ *(Rao, 1971).*

$$\mathbf{\Gamma}_{\mathcal{M}} \begin{pmatrix} \mathbf{G}' \\ \mathbf{L} \end{pmatrix} = \begin{pmatrix} \mathbf{V} & \mathbf{X} \\ \mathbf{X}' & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{G}' \\ \mathbf{L} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{X}' \end{pmatrix}. \tag{1.6}$$

Let

$$\mathbf{C}_{\mathcal{M}} = \begin{pmatrix} \mathbf{C}_{1\mathcal{M}} & \mathbf{C}_{2\mathcal{M}} \\ \mathbf{C}_{3\mathcal{M}} & -\mathbf{C}_{4\mathcal{M}} \end{pmatrix} = \begin{pmatrix} \mathbf{V} & \mathbf{X} \\ \mathbf{X}' & \mathbf{0} \end{pmatrix}^{-}, \tag{1.7}$$

where $\mathbf{C}_{1\mathcal{M}} \in \mathbb{R}^{n \times n}$ and $\mathbf{C}_{2\mathcal{M}} \in \mathbb{R}^{n \times p}$. Then

$$\begin{pmatrix} \mathbf{G}' \\ \mathbf{L} \end{pmatrix} = \begin{pmatrix} \mathbf{C}_{1\mathcal{M}} & \mathbf{C}_{2\mathcal{M}} \\ \mathbf{C}_{3\mathcal{M}} & -\mathbf{C}_{4\mathcal{M}} \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ \mathbf{X}' \end{pmatrix} = \begin{pmatrix} \mathbf{C}_{2\mathcal{M}}\mathbf{X}' \\ -\mathbf{C}_{4\mathcal{M}}\mathbf{X}' \end{pmatrix}. \tag{1.8}$$

and

$$\mathbf{G}\mathbf{y} = \mathbf{X}\mathbf{C}_{2\mathcal{M}}'\mathbf{y} \tag{1.9}$$

$$\text{BLUE}(\mathbf{X}\beta \mid \mathcal{M}) = \mathbf{X}\mathbf{C}'_{2\mathcal{M}}\mathbf{y}$$

From (1.8) we see that $\mathbf{X}\mathbf{C}'_{2\mathcal{M}}\mathbf{y}$ is one representation for the BLUE of $\mathbf{X}\beta$ under $\mathcal{M}$. Of course, if we let $\mathbf{C}_{\mathcal{M}}$ vary through all generalized inverses of $\mathbf{\Gamma}_{\mathcal{M}}$ we obtain all solutions to (1.6) and thereby all representations $\mathbf{G}\mathbf{y}$ for the BLUE of $\mathbf{X}\beta$ under $\mathcal{M}$; see Rao & Mitra (1971, p.29).

Furthermore, as a result of some matrix operations we obtain

$$\mathbf{X}\mathbf{C}'_{2\mathcal{M}}\mathbf{X} = \mathbf{X} \quad \text{and} \quad \mathbf{X}\mathbf{C}_{3\mathcal{M}}\mathbf{X} = \mathbf{X}, \tag{1.10}$$

$$\mathbf{V}\mathbf{C}_{2\mathcal{M}}\mathbf{X}' = \mathbf{X}\mathbf{C}'_{2\mathcal{M}}\mathbf{V} = \mathbf{V}\mathbf{C}'_{3\mathcal{M}}\mathbf{X}' = \mathbf{X}\mathbf{C}_{3\mathcal{M}}\mathbf{V}. \tag{1.11}$$

$$\text{BLUE}(\mathbf{X}\beta \mid \mathcal{M}) = \mathbf{X}\mathbf{C}'_{2\mathcal{M}}\mathbf{y}$$

From (1.8) we see that $\mathbf{X}\mathbf{C}'_{2\mathcal{M}}\mathbf{y}$ is one representation for the BLUE of $\mathbf{X}\beta$ under $\mathcal{M}$. Of course, if we let $\mathbf{C}_{\mathcal{M}}$ vary through all generalized inverses of $\mathbf{\Gamma}_{\mathcal{M}}$ we obtain all solutions to (1.6) and thereby all representations $\mathbf{G}\mathbf{y}$ for the BLUE of $\mathbf{X}\beta$ under $\mathcal{M}$; see Rao & Mitra (1971, p.29).

Furthermore, as a result of some matrix operations we obtain

$$\mathbf{X}\mathbf{C}'_{2\mathcal{M}}\mathbf{X} = \mathbf{X} \quad \text{and} \quad \mathbf{X}\mathbf{C}_{3\mathcal{M}}\mathbf{X} = \mathbf{X}, \tag{1.10}$$

$$\mathbf{V}\mathbf{C}_{2\mathcal{M}}\mathbf{X}' = \mathbf{X}\mathbf{C}'_{2\mathcal{M}}\mathbf{V} = \mathbf{V}\mathbf{C}'_{3\mathcal{M}}\mathbf{X}' = \mathbf{X}\mathbf{C}_{3\mathcal{M}}\mathbf{V}. \tag{1.11}$$

As further references to the Pandora's Box, we may mention

- Rao (1972a, 1972b, p.298–300, 1973a).
- Hall & Meyer (1975, Theorem 4.2) showed a property that the $\mathbf{C}_{i\mathcal{M}}$-matrices are independent of each other in the sense that if $\mathbf{D}_{1\mathcal{M}}$ is any $\mathbf{C}_{1\mathcal{M}}$-matrix and $\mathbf{D}_{2\mathcal{M}}$ is any $\mathbf{C}_{2\mathcal{M}}$-matrix etc., then the partitioned matrix composed by the $\mathbf{D}_{i\mathcal{M}}$-matrices is always a generalized inverse of $\mathbf{\Gamma}_{\mathcal{M}}$.
- Harville (1997, Sec 19.4).

For the computational aspects of the Pandora's Box, see

- Werner (1987).

Rao (1971) shows that the problem of inference from a linear model can be completely solved when one has obtained a matrix $\mathbf{C}_{\mathcal{M}} \in \{\mathbf{\Gamma}^{-}_{\mathcal{M}}\}$ and calls that the computation of the matrix $\mathbf{C}_{\mathcal{M}}$ is like opening a Pandora's Box, giving all that is necessary for drawing inferences on $\boldsymbol{\beta}$.

We may also cite Rao (1971, p.378):

> *"Once a generalized inverse is computed by a suitable procedure, we seem to have a Pandora Box supplying all the ingredients needed for obtaining the* BLUE*s, their variances and covariances, an unbiased estimator of σ^2, and test criteria without any further computations except for a few matrix multiplications...Thus the problem of inference from a linear model is reduced to numerical problem of finding an inverse (or generalized inverse) of the symmetric matrix* $\begin{pmatrix} \mathbf{V} & \mathbf{X} \\ \mathbf{X}' & \mathbf{0} \end{pmatrix}^{-} = \begin{pmatrix} \mathbf{C}_{1\mathcal{M}} & \mathbf{C}_{2\mathcal{M}} \\ \mathbf{C}_{3\mathcal{M}} & -\mathbf{C}_{4\mathcal{M}} \end{pmatrix}$*"*

Rao (1971) shows that the problem of inference from a linear model can be completely solved when one has obtained a matrix $\mathbf{C}_{\mathcal{M}} \in \{\mathbf{\Gamma}^{-}_{\mathcal{M}}\}$ and calls that the computation of the matrix $\mathbf{C}_{\mathcal{M}}$ is like opening a Pandora's Box, giving all that is necessary for drawing inferences on $\boldsymbol{\beta}$.

We may also cite Rao (1971, p.378):

> *"Once a generalized inverse is computed by a suitable procedure, we seem to have a Pandora Box supplying all the ingredients needed for obtaining the* BLUE*s, their variances and covariances, an unbiased estimator of σ^2, and test criteria without any further computations except for a few matrix multiplications...Thus the problem of inference from a linear model is reduced to numerical problem of finding an inverse (or generalized inverse) of the symmetric matrix $\begin{pmatrix} \mathbf{V} & \mathbf{X} \\ \mathbf{X}' & \mathbf{0} \end{pmatrix}^{-} = \begin{pmatrix} \mathbf{C}_{1\mathcal{M}} & \mathbf{C}_{2\mathcal{M}} \\ \mathbf{C}_{3\mathcal{M}} & -\mathbf{C}_{4\mathcal{M}} \end{pmatrix}$"*

The main purpose of this study is to give some results related to the equivalence between the BLUEs of $\mathbf{X}_1\boldsymbol{\beta}_1$ under two partitioned linear models

$$\mathcal{A} = \{\mathbf{y}, \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2, \mathbf{V}_{\mathcal{A}}\} \text{ and } \mathcal{B} = \{\mathbf{y}, \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2, \mathbf{V}_{\mathcal{B}}\},$$

which differ only in their covariance matrices.

There are many studies related to these topics, e.g.,

- Mitra & Moore (1973),
- Rao (1973b),
- Mathew & Bhimisankaram (1983),
- Baksalary & Mathew (1986),
- Haslett (1996),
- Werner & Yapar (1996),
- Isotalo, Puntanen & Styan (2007),
- Haslett & Puntanen (2010),
- Hauke, Markiewicz & Puntanen (2012).

The main purpose of this study is to give some results related to the equivalence between the BLUEs of $\mathbf{X}_1\beta_1$ under two partitioned linear models

$$\mathcal{A} = \{\mathbf{y}, \mathbf{X}_1\beta_1 + \mathbf{X}_2\beta_2, \mathbf{V}_{\mathcal{A}}\} \text{ and } \mathcal{B} = \{\mathbf{y}, \mathbf{X}_1\beta_1 + \mathbf{X}_2\beta_2, \mathbf{V}_{\mathcal{B}}\},$$

which differ only in their covariance matrices.

There are many studies related to these topics, e.g.,

- Mitra & Moore (1973),
- Rao (1973b),
- Mathew & Bhimisankaram (1983),
- Baksalary & Mathew (1986),
- Haslett (1996),
- Werner & Yapar (1996),
- Isotalo, Puntanen & Styan (2007),
- Haslett & Puntanen (2010),
- Hauke, Markiewicz & Puntanen (2012).

- Hauke, Markiewicz & Puntanen (2012) considered the Euclidean difference between the BLUEs under $\mathcal{A}$ and $\mathcal{B}$ and in particular utilise the concept of linear sufficiency in their considerations.
- Haslett & Puntanen (2010) gave a necessary and sufficient condition for the equality between the BLUEs of $\mathbf{X}_1\beta_1$ under two models which have different covariance matrices. They also considered the equality of the BLUEs under the full models assuming that they are equal under the submodels.

- Hauke, Markiewicz & Puntanen (2012) considered the Euclidean difference between the BLUEs under $\mathcal{A}$ and $\mathcal{B}$ and in particular utilise the concept of linear sufficiency in their considerations.
- Haslett & Puntanen (2010) gave a necessary and sufficient condition for the equality between the BLUEs of $\mathbf{X}_1\beta_1$ under two models which have different covariance matrices. They also considered the equality of the BLUEs under the full models assuming that they are equal under the submodels.

In our study, the problems studied in are reconsidered and in particular, the $\mathbf{C}_{i\mathcal{M}}$-matrices are used to give additional new results.

- Hauke, Markiewicz & Puntanen (2012) considered the Euclidean difference between the BLUEs under $\mathcal{A}$ and $\mathcal{B}$ and in particular utilise the concept of linear sufficiency in their considerations.
- Haslett & Puntanen (2010) gave a necessary and sufficient condition for the equality between the BLUEs of $\mathbf{X}_1\beta_1$ under two models which have different covariance matrices. They also considered the equality of the BLUEs under the full models assuming that they are equal under the submodels.

In our study, the problems studied in are reconsidered and in particular, the $\mathbf{C}_{i\mathcal{M}}$-matrices are used to give additional new results.

Furthermore, in Section 4, we briefly investigate the equality of the BLUEs of $\mathbf{FX}\beta$ under the model $\mathcal{M}$ and the transformed model

$$\mathcal{F} = \{\mathbf{Fy}, \mathbf{FX}, \mathbf{FVF}'\},$$

which is obtained from premultiplying the model $\mathcal{M} = \{\mathbf{y}, \mathbf{X}\beta, \mathbf{V}\}$ by an arbitrary non-zero matrix $\mathbf{F}$. For example, choosing $\mathbf{F} = \mathbf{M}_2$ we obtain the reduced model

$$\{\mathbf{M}_2\mathbf{y}, \mathbf{M}_2\mathbf{X}_1\beta_1, \mathbf{M}_2\mathbf{V}\mathbf{M}_2\}.$$

Furthermore, in Section 4, we briefly investigate the equality of the BLUEs of $\mathbf{FX}\beta$ under the model $\mathcal{M}$ and the transformed model

$$\mathcal{F} = \{\mathbf{Fy}, \mathbf{FX}, \mathbf{FVF}'\},$$

which is obtained from premultiplying the model $\mathcal{M} = \{\mathbf{y}, \mathbf{X}\beta, \mathbf{V}\}$ by an arbitrary non-zero matrix $\mathbf{F}$. For example, choosing $\mathbf{F} = \mathbf{M}_2$ we obtain the reduced model

$$\{\mathbf{M}_2\mathbf{y}, \mathbf{M}_2\mathbf{X}_1\beta_1, \mathbf{M}_2\mathbf{V}\mathbf{M}_2\}.$$

Our considerations are based on the Pandora's Box approach.

Furthermore, in Section 4, we briefly investigate the equality of the BLUEs of $\mathbf{FX}\beta$ under the model $\mathcal{M}$ and the transformed model

$$\mathcal{F} = \{\mathbf{Fy}, \mathbf{FX}, \mathbf{FVF'}\},$$

which is obtained from premultiplying the model $\mathcal{M} = \{\mathbf{y}, \mathbf{X}\beta, \mathbf{V}\}$ by an arbitrary non-zero matrix $\mathbf{F}$. For example, choosing $\mathbf{F} = \mathbf{M}_2$ we obtain the reduced model

$$\{\mathbf{M}_2\mathbf{y}, \mathbf{M}_2\mathbf{X}_1\beta_1, \mathbf{M}_2\mathbf{V}\mathbf{M}_2\}.$$

Our considerations are based on the Pandora's Box approach.

- Farebrother (1979),
- Baksalary & Kala (1981),
- Stahlecker & Schmidt (1987),
- Lucke (1991),
- Gross & Trenkler (1997),
- Gross, Trenkler & Werner (2001),
- Zhang, Liu & Lu (2004),
- Zhang (2007),
- Tian & Puntanen (2009).

2. Two Partitioned Linear Models With Different Covariance Matrices

Consider two partitioned linear models

$$\mathcal{A} = \{\mathbf{y}, \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2, \mathbf{V}_{\mathcal{A}}\} \text{ and } \mathcal{B} = \{\mathbf{y}, \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2, \mathbf{V}_{\mathcal{B}}\}, \tag{2.1}$$

with their submodels

$$\mathcal{A}_i = \{\mathbf{y}, \mathbf{X}_i\boldsymbol{\beta}_i, \mathbf{V}_{\mathcal{A}}\}, \quad \mathcal{B}_i = \{\mathbf{y}, \mathbf{X}_i\boldsymbol{\beta}_i, \mathbf{V}_{\mathcal{B}}\}, \quad i = 1, 2, \tag{2.2}$$

and the corresponding reduced models

$$\mathcal{A}_r = \{\mathbf{M}_2\mathbf{y}, \mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1, \mathbf{M}_2\mathbf{V}_{\mathcal{A}}\mathbf{M}_2\}, \tag{2.3}$$

$$\mathcal{B}_r = \{\mathbf{M}_2\mathbf{y}, \mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1, \mathbf{M}_2\mathbf{V}_{\mathcal{B}}\mathbf{M}_2\}. \tag{2.4}$$

Suppose that every representation of $\mathrm{BLUE}(\mathbf{X}_1\beta_1)$ under $\mathcal{A}_1$ continues to be $\mathrm{BLUE}(\mathbf{X}_1\beta_1)$ under $\mathcal{B}_1$ or, in short,

$$\{\mathrm{BLUE}(\mathbf{X}_1\beta_1 \mid \mathcal{A}_1)\} \subseteq \{\mathrm{BLUE}(\mathbf{X}_1\beta_1 \mid \mathcal{B}_1)\}. \tag{2.5}$$

In matrix terms, the notation in (2.5) means that

$$\mathbf{G} \text{ satisfies } \mathbf{G}(\mathbf{X}_1 : \mathbf{V}_{\mathcal{A}}\mathbf{M}_1) = (\mathbf{X}_1 : \mathbf{0}) \tag{2.6a}$$

$$\Longrightarrow$$

$$\mathbf{G}(\mathbf{X}_1 : \mathbf{V}_{\mathcal{B}}\mathbf{M}_1) = (\mathbf{X}_1 : \mathbf{0}). \tag{2.6b}$$

According to Lemma 1.1, the BLUE of $\mathbf{X}_1\boldsymbol{\beta}_1$ under $\mathcal{A}_1$ can be expressed as

$$\mathrm{BLUE}(\mathbf{X}_1\boldsymbol{\beta}_1 \mid \mathcal{A}_1) = \mathbf{X}_1\mathbf{C}'_{2\mathcal{A}_1}\mathbf{y}, \tag{2.7}$$

where $\mathbf{C}_{2\mathcal{A}_1} \in \mathbb{R}^{n\times p_1}$ is the 12-block of the matrix

$$\mathbf{C}_{\mathcal{A}_1} = \begin{pmatrix} \mathbf{C}_{1\mathcal{A}_1} & \mathbf{C}_{2\mathcal{A}_1} \\ \mathbf{C}_{3\mathcal{A}_1} & -\mathbf{C}_{4\mathcal{A}_1} \end{pmatrix} = \begin{pmatrix} \mathbf{V}_{\mathcal{A}_1} & \mathbf{X}_1 \\ \mathbf{X}'_1 & \mathbf{0} \end{pmatrix}^{-} \in \{\boldsymbol{\Gamma}^{-}_{\mathcal{A}_1}\}, \tag{2.8}$$

obtained for the model $\mathcal{A}_1$. Hence,
$\{\mathrm{BLUE}(\mathbf{X}_1\boldsymbol{\beta}_1 \mid \mathcal{A}_1)\} \subseteq \{\mathrm{BLUE}(\mathbf{X}_1\boldsymbol{\beta}_1 \mid \mathcal{B}_1)\}$ can be equivalently expressed as

$$\mathbf{X}_1\mathbf{C}'_{2\mathcal{A}_1}\mathbf{y} \text{ is BLUE for } \mathbf{X}_1\boldsymbol{\beta}_1 \text{ under } \mathcal{B}_1 \text{ for every } \mathbf{C}_{2\mathcal{A}_1}. \tag{2.9}$$

Because $\mathbf{X}_1\mathbf{C}'_{2\mathcal{A}_1}\mathbf{X}_1 = \mathbf{X}_1$, (2.5) is equivalent to $\mathbf{X}_1\mathbf{C}'_{2\mathcal{A}_1}\mathbf{V}_{\mathcal{B}}\mathbf{M}_1 = \mathbf{0}$.

In the following lemma we collect some conditions for (2.5).

Lemma (2.1)

Consider the submodels $\mathcal{A}_1 = \{\mathbf{y}, \mathbf{X}_1\boldsymbol{\beta}_1, \mathbf{V}_\mathcal{A}\}$ *and* $\mathcal{B}_1 = \{\mathbf{y}, \mathbf{X}_1\boldsymbol{\beta}_1, \mathbf{V}_\mathcal{B}\}$*. Then the following statements are equivalent:*

(a) $\{\mathrm{BLUE}(\mathbf{X}_1\boldsymbol{\beta}_1 \mid \mathcal{A}_1)\} \subseteq \{\mathrm{BLUE}(\mathbf{X}_1\boldsymbol{\beta}_1 \mid \mathcal{B}_1)\}$.

(b) $\mathscr{C}(\mathbf{V}_\mathcal{B}\mathbf{X}_1^\perp) \subseteq \mathscr{C}(\mathbf{V}_\mathcal{A}\mathbf{X}_1^\perp)$, *i.e.,* $\mathscr{C}(\mathbf{V}_\mathcal{B}\mathbf{M}_1) \subseteq \mathscr{C}(\mathbf{V}_\mathcal{A}\mathbf{M}_1)$.

(c) $\mathbf{X}_1\mathbf{C}'_{2\mathcal{A}_1}\mathbf{V}_\mathcal{B}\mathbf{M}_1 = \mathbf{0}$ *for every* $\mathbf{C}_{2\mathcal{A}_1} \in \{(\boldsymbol{\Gamma}^-_{\mathcal{A}_1})_{12}\}$.

(d) $\mathscr{C}(\mathbf{C}'_{2\mathcal{A}_1}\mathbf{V}_\mathcal{B}\mathbf{M}_1) \subseteq \mathscr{C}(\mathbf{C}'_{2\mathcal{A}_1}\mathbf{V}_\mathcal{A}\mathbf{M}_1)$ *for every* $\mathbf{C}_{2\mathcal{A}_1} \in \{(\boldsymbol{\Gamma}^-_{\mathcal{A}_1})_{12}\}$.

Proof.

The equivalence of (a) and (b) is shown, e.g., in

- Mitra & Moore (1973, Theorem 4.1–4.2),
- Rao (1973b, p.289),
- Baksalary & Mathew (1986, Theorem 3).

The equivalence of (a) and (c) is obvious.
The inclusion (b) trivially implies (d).
Suppose that (d) holds. Then there exists a matrix $\mathbf{K}$ such that

$$\mathbf{C}'_{2\mathcal{A}_1}\mathbf{V}_{\mathcal{B}}\mathbf{M}_1 = \mathbf{C}'_{2\mathcal{A}_1}\mathbf{V}_{\mathcal{A}}\mathbf{M}_1\mathbf{K}. \tag{2.10}$$

Premultiplying (2.10) by $\mathbf{X}_1$ gives

$$\mathbf{X}_1\mathbf{C}'_{2\mathcal{A}_1}\mathbf{V}_{\mathcal{B}}\mathbf{M}_1 = \mathbf{X}_1\mathbf{C}'_{2\mathcal{A}_1}\mathbf{V}_{\mathcal{A}}\mathbf{M}_1\mathbf{K} = \mathbf{0}. \tag{2.11}$$

Thus (a) is equivalent to (d). □

In the following theorem, we consider the BLUEs of $\mathbf{X}_1\beta_1$ under the full models $\mathcal{A}$ and $\mathcal{B}$.

Theorem (2.1)

Consider the partitioned linear models $\mathcal{A} = \{\mathbf{y}, \mathbf{X}_1\beta_1 + \mathbf{X}_2\beta_2, \mathbf{V}_{\mathcal{A}}\}$ *and* $\mathcal{B} = \{\mathbf{y}, \mathbf{X}_1\beta_1 + \mathbf{X}_2\beta_2, \mathbf{V}_{\mathcal{B}}\}$ *and assume that* $\mathbf{X}_1\beta_1$ *is estimable under* $\mathcal{A}$. *Then the following statements are equivalent:*

(a) $\{\mathrm{BLUE}(\mathbf{X}_1\beta_1 \mid \mathcal{A})\} \subseteq \{\mathrm{BLUE}(\mathbf{X}_1\beta_1 \mid \mathcal{B})\}$.

(b) $\mathscr{C}[\mathbf{M}_2\mathbf{V}_{\mathcal{B}}\mathbf{M}_2(\mathbf{M}_2\mathbf{X}_1)^{\perp}] \subseteq \mathscr{C}[\mathbf{M}_2\mathbf{V}_{\mathcal{A}}\mathbf{M}_2(\mathbf{M}_2\mathbf{X}_1)^{\perp}]$.

(c) $\mathscr{C}(\mathbf{M}_2\mathbf{V}_{\mathcal{B}}\mathbf{M}) \subseteq \mathscr{C}(\mathbf{M}_2\mathbf{V}_{\mathcal{A}}\mathbf{M})$.

(d) $\mathbf{X}_1\mathbf{C}'_{2\mathcal{A}_r}\mathbf{M}_2\mathbf{V}_{\mathcal{B}}\mathbf{M} = \mathbf{0}$ *for every* $\mathbf{C}_{2\mathcal{A}_r} \in \{(\mathbf{\Gamma}^{-}_{\mathcal{A}_r})_{12}\}$.

(e) $\mathscr{C}(\mathbf{C}'_{2\mathcal{A}_r}\mathbf{M}_2\mathbf{V}_{\mathcal{B}}\mathbf{M}) \subseteq \mathscr{C}(\mathbf{C}'_{2\mathcal{A}_r}\mathbf{M}_2\mathbf{V}_{\mathcal{A}}\mathbf{M})$ *for every* $\mathbf{C}_{2\mathcal{A}_r} \in \{(\mathbf{\Gamma}^{-}_{\mathcal{A}_r})_{12}\}$.

Here $\mathbf{C}_{2\mathcal{A}_r}$ *refers to the* 12*-block of a generalized inverse of the matrix*

$$\mathbf{\Gamma}_{\mathcal{A}_r} = \begin{pmatrix} \mathbf{M}_2\mathbf{V}_{\mathcal{A}}\mathbf{M}_2 & \mathbf{M}_2\mathbf{X}_1 \\ \mathbf{X}_1'\mathbf{M}_2 & \mathbf{0} \end{pmatrix}. \tag{2.12}$$

In the following theorem, we consider the BLUEs of $\mathbf{X}_1\boldsymbol{\beta}_1$ under the full models $\mathcal{A}$ and $\mathcal{B}$.

Theorem (2.1)

Consider the partitioned linear models $\mathcal{A} = \{\mathbf{y}, \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2, \mathbf{V}_\mathcal{A}\}$ *and* $\mathcal{B} = \{\mathbf{y}, \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2, \mathbf{V}_\mathcal{B}\}$ *and assume that* $\mathbf{X}_1\boldsymbol{\beta}_1$ *is estimable under* $\mathcal{A}$. *Then the following statements are equivalent:*

(a) $\{\mathrm{BLUE}(\mathbf{X}_1\boldsymbol{\beta}_1 \mid \mathcal{A})\} \subseteq \{\mathrm{BLUE}(\mathbf{X}_1\boldsymbol{\beta}_1 \mid \mathcal{B})\}$.

(b) $\mathscr{C}[\mathbf{M}_2\mathbf{V}_\mathcal{B}\mathbf{M}_2(\mathbf{M}_2\mathbf{X}_1)^\perp] \subseteq \mathscr{C}[\mathbf{M}_2\mathbf{V}_\mathcal{A}\mathbf{M}_2(\mathbf{M}_2\mathbf{X}_1)^\perp]$.

(c) $\mathscr{C}(\mathbf{M}_2\mathbf{V}_\mathcal{B}\dot{\mathbf{M}}) \subseteq \mathscr{C}(\mathbf{M}_2\mathbf{V}_\mathcal{A}\dot{\mathbf{M}})$.

(d) $\mathbf{X}_1\mathbf{C}'_{2\mathcal{A}_r}\mathbf{M}_2\mathbf{V}_\mathcal{B}\dot{\mathbf{M}} = \mathbf{0}$ *for every* $\mathbf{C}_{2\mathcal{A}_r} \in \{(\boldsymbol{\Gamma}^-_{\mathcal{A}_r})_{12}\}$.

(e) $\mathscr{C}(\mathbf{C}'_{2\mathcal{A}_r}\mathbf{M}_2\mathbf{V}_\mathcal{B}\dot{\mathbf{M}}) \subseteq \mathscr{C}(\mathbf{C}'_{2\mathcal{A}_r}\mathbf{M}_2\mathbf{V}_\mathcal{A}\dot{\mathbf{M}})$ *for every* $\mathbf{C}_{2\mathcal{A}_r} \in \{(\boldsymbol{\Gamma}^-_{\mathcal{A}_r})_{12}\}$.

Here $\mathbf{C}_{2\mathcal{A}_r}$ *refers to the* 12*-block of a generalized inverse of the matrix*

$$\boldsymbol{\Gamma}_{\mathcal{A}_r} = \begin{pmatrix} \mathbf{M}_2\mathbf{V}_\mathcal{A}\mathbf{M}_2 & \mathbf{M}_2\mathbf{X}_1 \\ \mathbf{X}_1'\mathbf{M}_2 & \mathbf{0} \end{pmatrix}. \tag{2.12}$$

Proof.

The proof for the equivalence of (a), (b) and (c) appears in

- Mathew & Bhimasankaram (1983, Theorem 2.4),
- Haslett & Puntanen (2010, Theorem 2.1),
- Puntanen, Styan & Isotalo (2011, Sec.15.6).

Notice that

- $\mathbf{M} = \mathbf{I}_n - \mathbf{P}_{(\mathbf{X}_1 : \mathbf{X}_2)} = \mathbf{I}_n - \mathbf{P}_{\mathbf{X}_2} - \mathbf{P}_{\mathbf{M}_2\mathbf{X}_1} = \mathbf{M}_2\mathbf{Q}_{\mathbf{M}_2\mathbf{X}_1}$.
- $\mathbf{Q}_{\mathbf{M}_2\mathbf{X}_1} = \mathbf{I}_n - \mathbf{P}_{\mathbf{M}_2\mathbf{X}_1}$ is one choice for $(\mathbf{M}_2\mathbf{X}_1)^{\perp}$.

□

Proof.

The proof can be based on the generalized Frisch–Waugh–Lovell theorem (Gross & Puntanen, 2000, Theorem 4), which states that every BLUE for estimable $\mathbf{X}_1\beta_1$ under the partitioned model $\mathcal{A}$ remains BLUE for $\mathbf{X}_1\beta_1$ under the reduced model $\mathcal{A}_r$ and vice versa. Hence the comparison of the BLUEs for $\mathbf{X}_1\beta_1$ under the full models $\mathcal{A}$ and $\mathcal{B}$ can be considered as the corresponding problem under the reduced models $\mathcal{A}_r$ and $\mathcal{B}_r$.

To consider (d), we know that $\mathbf{M}_2\mathbf{X}_1\mathbf{C}'_{2\mathcal{A}_r}\mathbf{M}_2\mathbf{y}$ is the BLUE for $\mathbf{M}_2\mathbf{X}_1\beta_1$ under $\mathcal{A}_r$ and moreover, it is confirmed that $\mathbf{X}_1\mathbf{C}'_{2\mathcal{A}_r}\mathbf{M}_2\mathbf{y}$ is the BLUE for $\mathbf{X}_1\beta_1$ under $\mathcal{A}_r$. Therefore, (a) means that for every $\mathbf{C}_{2\mathcal{A}_r} \in \{(\mathbf{\Gamma}^-_{\mathcal{A}_r})_{12}\}$, $\mathbf{X}_1\mathbf{C}'_{2\mathcal{A}_r}\mathbf{M}_2\mathbf{y}$ continues to be the BLUE under $\mathcal{B}_r$, i.e.,

$$\mathbf{X}_1\mathbf{C}'_{2\mathcal{A}_r}\mathbf{M}_2\mathbf{V}_{\mathcal{B}}\mathbf{M}_2\mathbf{Q}_{\mathbf{M}_2\mathbf{X}_1} = \mathbf{0} \quad \text{for every } \mathbf{C}_{2\mathcal{A}_r} \in \{(\mathbf{\Gamma}^-_{\mathcal{A}_r})_{12}\}, \tag{2.13}$$

and thereby the equivalence of (a) and (d) is confirmed.

The equivalence of (a) and (e) can be concluded from Lemma 2.1. □

3. The Equality of BLUEs Under the Full Models When They Equal Under the Submodels

In this section, the equality of BLUE($\mathbf{X}_1\beta_1$) under the models $\mathcal{A}$ and $\mathcal{B}$ is considered when it is assumed that the BLUEs coincide under the submodels $\mathcal{A}_1$ and $\mathcal{B}_1$. We will assume that

$$\mathscr{C}(\mathbf{X} : \mathbf{V}_{\mathcal{A}}) = \mathscr{C}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{V}_{\mathcal{A}}) = \mathscr{C}(\mathbf{X}_1 : \mathbf{V}_{\mathcal{A}}) \qquad (3.1)$$

to avoid any contradictions since the models $\mathcal{A}_1$ and $\mathcal{A}$ are considered at the same time; see Haslett & Puntanen (2010, p. 108). The assumption (3.1) means that

$$\mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{X}_1 : \mathbf{V}_{\mathcal{A}}) = \mathscr{C}(\mathbf{X}_1 : \mathbf{V}_{\mathcal{A}}\mathbf{M}_1). \qquad (3.2)$$

Let us denote $\mathbf{G}_i\mathbf{y} = \mathrm{BLUE}(\mathbf{X}_i\boldsymbol{\beta}_i)$ under $\mathcal{A}$, $i = 1, 2$, and $\mathbf{G}_3\mathbf{y} = \mathrm{BLUE}(\mathbf{X}_1\boldsymbol{\beta}_1)$ under $\mathcal{A}_1$, so that $\mathbf{G}_1$, $\mathbf{G}_2$ and $\mathbf{G}_3$ are any matrices satisfying the equations

$$\begin{pmatrix} \mathbf{G}_1 \\ \mathbf{G}_2 \end{pmatrix} (\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{V}_{\mathcal{A}}\mathbf{M}) = \begin{pmatrix} \mathbf{X}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_2 & \mathbf{0} \end{pmatrix}, \tag{3.3a}$$

$$\mathbf{G}_3(\mathbf{X}_1 : \mathbf{V}_{\mathcal{A}}\mathbf{M}_1) = (\mathbf{X}_1 : \mathbf{0}). \tag{3.3b}$$

In the following lemma, we represent an updating formula for the BLUE$(\mathbf{X}_1\beta_1 \mid \mathcal{A})$.

Lemma (3.1)

Consider the partitioned linear model $\mathcal{A}$ and assume that $\mathbf{X}_1\beta_1$ is estimable, $\mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{X}_1 : \mathbf{V}_{\mathcal{A}})$, and let the matrix $\mathbf{W}_1$ be defined as $\mathbf{W}_1 = \mathbf{V}_{\mathcal{A}} + \mathbf{X}_1\mathbf{U}\mathbf{X}_1'$ such that $\mathscr{C}(\mathbf{W}_1) = \mathscr{C}(\mathbf{X}_1 : \mathbf{V}_{\mathcal{A}})$. Then

$$\begin{aligned} \mathbf{X}_1\tilde{\beta}_1(\mathcal{A}) &= \mathbf{X}_1\tilde{\beta}_1(\mathcal{A}_1) - \mathbf{X}_1(\mathbf{X}_1'\mathbf{W}_1^-\mathbf{X}_1)^-\mathbf{X}_1'\mathbf{W}_1^-\mathbf{X}_2\tilde{\beta}_2(\mathcal{A}) \\ &= \mathbf{X}_1\tilde{\beta}_1(\mathcal{A}_1) - \mathbf{X}_1\mathbf{Z}\mathbf{X}_2\tilde{\beta}_2(\mathcal{A}), \end{aligned} \tag{3.4}$$

where $\mathbf{Z} = (\mathbf{X}_1'\mathbf{W}_1^-\mathbf{X}_1)^-\mathbf{X}_1'\mathbf{W}_1^-$, or in other notation, with $\mathbf{G}_1$, $\mathbf{G}_2$ and $\mathbf{G}_3$ being defined as in (3.3),

$$\mathbf{G}_1\mathbf{y} = \mathbf{G}_3\mathbf{y} - \mathbf{X}_1\mathbf{Z}\mathbf{G}_2\mathbf{y} \quad \text{for all } \mathbf{y} \in \mathscr{C}(\mathbf{X} : \mathbf{V}_{\mathcal{A}}\mathbf{M}), \tag{3.5}$$

or, equivalently, in terms of Pandora's Box,

$$\mathbf{X}_1\tilde{\beta}_1(\mathcal{A}) = \mathbf{X}_1\mathbf{C}_{2\mathcal{A}_1}'\mathbf{y} - \mathbf{X}_1\mathbf{C}_{2\mathcal{A}_1}'\mathbf{X}_2\tilde{\beta}_2(\mathcal{A}). \tag{3.6}$$

In the following lemma, we represent an updating formula for the $\mathrm{BLUE}(\mathbf{X}_1\boldsymbol{\beta}_1 \mid \mathcal{A})$.

Lemma (3.1)

Consider the partitioned linear model $\mathcal{A}$ and assume that $\mathbf{X}_1\boldsymbol{\beta}_1$ is estimable, $\mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{X}_1 : \mathbf{V}_{\mathcal{A}})$, and let the matrix $\mathbf{W}_1$ be defined as $\mathbf{W}_1 = \mathbf{V}_{\mathcal{A}} + \mathbf{X}_1\mathbf{U}\mathbf{X}_1'$ such that $\mathscr{C}(\mathbf{W}_1) = \mathscr{C}(\mathbf{X}_1 : \mathbf{V}_{\mathcal{A}})$. Then

$$\begin{aligned} \mathbf{X}_1\tilde{\boldsymbol{\beta}}_1(\mathcal{A}) &= \mathbf{X}_1\tilde{\boldsymbol{\beta}}_1(\mathcal{A}_1) - \mathbf{X}_1(\mathbf{X}_1'\mathbf{W}_1^-\mathbf{X}_1)^-\mathbf{X}_1'\mathbf{W}_1^-\mathbf{X}_2\tilde{\boldsymbol{\beta}}_2(\mathcal{A}) \\ &= \mathbf{X}_1\tilde{\boldsymbol{\beta}}_1(\mathcal{A}_1) - \mathbf{X}_1\mathbf{Z}\mathbf{X}_2\tilde{\boldsymbol{\beta}}_2(\mathcal{A}), \end{aligned} \tag{3.4}$$

where $\mathbf{Z} = (\mathbf{X}_1'\mathbf{W}_1^-\mathbf{X}_1)^-\mathbf{X}_1'\mathbf{W}_1^-$, or in other notation, with $\mathbf{G}_1$, $\mathbf{G}_2$ and $\mathbf{G}_3$ being defined as in (3.3)*,*

$$\mathbf{G}_1\mathbf{y} = \mathbf{G}_3\mathbf{y} - \mathbf{X}_1\mathbf{Z}\mathbf{G}_2\mathbf{y} \quad \textit{for all } \mathbf{y} \in \mathscr{C}(\mathbf{X} : \mathbf{V}_{\mathcal{A}}\mathbf{M}), \tag{3.5}$$

or, equivalently, in terms of Pandora's Box,

$$\mathbf{X}_1\tilde{\boldsymbol{\beta}}_1(\mathcal{A}) = \mathbf{X}_1\mathbf{C}_{2\mathcal{A}_1}'\mathbf{y} - \mathbf{X}_1\mathbf{C}_{2\mathcal{A}_1}'\mathbf{X}_2\tilde{\boldsymbol{\beta}}_2(\mathcal{A}). \tag{3.6}$$

In this lemma, $\tilde{\beta}_1(\mathcal{A})$ refers to any vector $\mathbf{Ly}$ such that $\mathbf{X}_1\tilde{\beta}_1 = \mathbf{X}_1\mathbf{Ly}$ is the BLUE for $\mathbf{X}_1\beta_1$ under $\mathcal{A}$.

Parts (3.4) and (3.5) of Lemma 3.1 appear in

- Werner & Yapar (1996, Theorem 2.3),
- Haslett & Puntanen (2010, Lemma 3.1),
- Haslett (1996): in the situation when $\mathbf{X}$ has full column rank and $\mathbf{V}_{\mathcal{A}}$ is positive definite.

The proof of part (3.6) of Lemma 3.1 is straightforward.

In the following theorem, we consider the equality of the BLUEs under the full models assuming that they are equal under the submodels.

Theorem (3.1)

Consider the partitioned linear models $\mathcal{A} = \{\mathbf{y}, \mathbf{X}_1\beta_1 + \mathbf{X}_2\beta_2, \mathbf{V}_{\mathcal{A}}\}$ *and* $\mathcal{B} = \{\mathbf{y}, \mathbf{X}_1\beta_1 + \mathbf{X}_2\beta_2, \mathbf{V}_{\mathcal{B}}\}$ *and assume that* $\mathbf{X}_1\beta_1$ *is estimable,* $\mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{X}_1 : \mathbf{V}_{\mathcal{A}})$, *and* $\mathbf{W}_1$ *is defined as in Lemma 4. Moreover, suppose that every representation of* $\mathbf{X}_1\beta_1$ *under* $\mathcal{A}_1$ *continues to be* BLUE *of* $\mathbf{X}_1\beta_1$ *under* $\mathcal{B}_1$ *and vice versa. Then the following statements are equivalent:*

(a) $\mathbf{X}_1\tilde{\beta}_1(\mathcal{A}) = \mathbf{X}_1\tilde{\beta}_1(\mathcal{B})$,

(b) $\mathbf{X}_1'\mathbf{W}_1^{-}\mathbf{X}_2[\tilde{\beta}_2(\mathcal{A}) - \tilde{\beta}_2(\mathcal{B})] = \mathbf{0}$.

(c) $\mathbf{X}_1\mathbf{C}_{2\mathcal{A}_1}'\mathbf{X}_2[\tilde{\beta}_2(\mathcal{A}) - \tilde{\beta}_2(\mathcal{B})] = \mathbf{0}$.

In the following theorem, we consider the equality of the BLUEs under the full models assuming that they are equal under the submodels.

Theorem (3.1)

Consider the partitioned linear models $\mathcal{A} = \{\mathbf{y}, \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2, \mathbf{V}_\mathcal{A}\}$ *and* $\mathcal{B} = \{\mathbf{y}, \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2, \mathbf{V}_\mathcal{B}\}$ *and assume that* $\mathbf{X}_1\boldsymbol{\beta}_1$ *is estimable,* $\mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{X}_1 : \mathbf{V}_\mathcal{A})$*, and* $\mathbf{W}_1$ *is defined as in Lemma 4. Moreover, suppose that every representation of* $\mathbf{X}_1\boldsymbol{\beta}_1$ *under* $\mathcal{A}_1$ *continues to be* BLUE *of* $\mathbf{X}_1\boldsymbol{\beta}_1$ *under* $\mathcal{B}_1$ *and vice versa. Then the following statements are equivalent:*

(a) $\mathbf{X}_1\tilde{\boldsymbol{\beta}}_1(\mathcal{A}) = \mathbf{X}_1\tilde{\boldsymbol{\beta}}_1(\mathcal{B})$,

(b) $\mathbf{X}_1'\mathbf{W}_1^-\mathbf{X}_2[\tilde{\boldsymbol{\beta}}_2(\mathcal{A}) - \tilde{\boldsymbol{\beta}}_2(\mathcal{B})] = \mathbf{0}$.

(c) $\mathbf{X}_1\mathbf{C}_{2\mathcal{A}_1}'\mathbf{X}_2[\tilde{\boldsymbol{\beta}}_2(\mathcal{A}) - \tilde{\boldsymbol{\beta}}_2(\mathcal{B})] = \mathbf{0}$.

Proof.

Haslett & Puntanen (2010, Theorem 3.1) proved the equivalence of (a) and (b). For related results, see

- Nurhonen & Puntanen (1992),
- Isotalo, Puntanen & Styan (2007): they studied whether the equality of the ordinary least squares estimator and the BLUE continues after elimination of one observation.

□

Proof.

To prove the equivalence of (a) and (c), we first write the decompositions

$$\mathbf{X}_1\tilde{\beta}_1(\mathcal{A}) = \mathbf{X}_1\mathbf{C}'_{2\mathcal{A}_1}\mathbf{y} - \mathbf{X}_1\mathbf{C}'_{2\mathcal{A}_1}\mathbf{X}_2\tilde{\beta}_2(\mathcal{A}), \tag{3.7}$$

$$\mathbf{X}_1\tilde{\beta}_1(\mathcal{B}) = \mathbf{X}_1\mathbf{C}'_{2\mathcal{B}_1}\mathbf{y} - \mathbf{X}_1\mathbf{C}'_{2\mathcal{B}_1}\mathbf{X}_2\tilde{\beta}_2(\mathcal{B}). \tag{3.8}$$

The assumption $\{\mathrm{BLUE}(\mathbf{X}_1\beta_1 \mid \mathcal{A}_1)\} = \{\mathrm{BLUE}(\mathbf{X}_1\beta_1 \mid \mathcal{B}_1)\}$ indicates that $\mathscr{C}(\mathbf{V}_\mathcal{B}\mathbf{M}_1) = \mathscr{C}(\mathbf{V}_\mathcal{A}\mathbf{M}_1)$, and hence the non-contradictory requirement under $\mathcal{A}$ in (3.2),

$$\mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{X}_1 : \mathbf{V}_\mathcal{A}\mathbf{M}_1), \text{ yields } \mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{X}_1 : \mathbf{V}_\mathcal{B}\mathbf{M}_1). \tag{3.9}$$

□

Proof.

Thus we have no contradictory problem in decomposition (3.8). By assumption, in (3.7)–(3.8) we have $\mathbf{X}_1\mathbf{C}'_{2\mathcal{A}_1}\mathbf{y} = \mathbf{X}_1\mathbf{C}'_{2\mathcal{B}_1}\mathbf{y}$ for all $\mathbf{y} \in \mathscr{C}(\mathbf{X}_1 : \mathbf{V}_{\mathcal{B}}\mathbf{M}_1)$. It remains to show that

$$\mathbf{X}_1\mathbf{C}'_{2\mathcal{A}_1}\mathbf{X}_2 = \mathbf{X}_1\mathbf{C}'_{2\mathcal{B}_1}\mathbf{X}_2. \tag{3.10}$$

In view of (3.9), there exist matrices $\mathbf{L}_1$, $\mathbf{L}_2$ and $\mathbf{L}_3$ so that

$$\mathbf{X}_2 = \mathbf{X}_1\mathbf{L}_1 + \mathbf{V}_{\mathcal{B}}\mathbf{M}_1\mathbf{L}_2 = \mathbf{X}_1\mathbf{L}_1 + \mathbf{V}_{\mathcal{A}}\mathbf{M}_1\mathbf{L}_3. \tag{3.11}$$

Substituting (3.11) into (3.10) confirms that (3.10) indeed holds. This completes the proof. □

4. The Equality of BLUEs Under the Full Model and the Transformed Model

Suppose that the matrix equation $\mathbf{G}(\mathbf{X} : \mathbf{VM}) = (\mathbf{X} : \mathbf{0})$ is premultiplied by the matrix $\mathbf{F}$ yielding $\mathbf{FG}(\mathbf{X} : \mathbf{VM}) = (\mathbf{FX} : \mathbf{0})$. This confirms that if $\mathbf{Gy}$ is the BLUE of $\mathbf{X}\beta$ under $\mathcal{M} = \{\mathbf{y}, \mathbf{X}\beta, \mathbf{V}\}$, then $\mathbf{FGy}$ is the BLUE of $\mathbf{FX}\beta$ under $\mathcal{M}$. Therefore, the BLUE of $\mathbf{FX}\beta$ under the model $\mathcal{M}$ can be expressed as

$$\text{BLUE}(\mathbf{FX}\beta \mid \mathcal{M}) = \mathbf{FXC}'_{2\mathcal{M}}\mathbf{y}. \tag{4.1}$$

Denoting

$$\mathbf{C}_{\mathcal{F}} = \begin{pmatrix} \mathbf{C}_{1\mathcal{F}} & \mathbf{C}_{2\mathcal{F}} \\ \mathbf{C}_{3\mathcal{F}} & -\mathbf{C}_{4\mathcal{F}} \end{pmatrix} \in \left\{ \begin{pmatrix} \mathbf{FVF}' & \mathbf{FX} \\ \mathbf{X}'\mathbf{F}' & \mathbf{0} \end{pmatrix}^{-} \right\} = \{\mathbf{\Gamma}_{\mathcal{F}}^{-}\} \,, \tag{4.2}$$

we get

$$\mathrm{BLUE}(\mathbf{FX}\beta \mid \mathcal{F}) = \mathbf{FXC}'_{2\mathcal{F}}\mathbf{Fy}. \tag{4.3}$$

Requesting the equality between (4.1) and (4.3) to hold for all $\mathbf{y} \in \mathscr{C}(\mathbf{X} : \mathbf{VM})$ means that

$$\mathbf{FXC}'_{2\mathcal{F}}\mathbf{FX} = \mathbf{FXC}'_{2\mathcal{M}}\mathbf{X}, \quad (4.4a)$$

$$\mathbf{FXC}'_{2\mathcal{F}}\mathbf{FVM} = \mathbf{FXC}'_{2\mathcal{M}}\mathbf{VM}. \quad (4.4b)$$

Equation (4.4a) trivially holds as each side equals **FX.** The right-hand side of (4.4b) is the null matrix and hence the equality between (4.1) and (4.3) holds (with probability one) if and only if

$$\mathbf{FXC}'_{2\mathcal{F}}\mathbf{FVM} = \mathbf{0}, \quad (4.5)$$

or equivalently,

$$\mathscr{C}(\mathbf{VM}) \subseteq \mathscr{N}(\mathbf{FXC}'_{2\mathcal{F}}\mathbf{F}). \quad (4.6)$$

5. Concluding Remarks

In this paper we have studied some problems related to linear statistical models using the inverse partitioned matrix (IPM) method, i.e., we are interested in $\mathbf{C}_{\mathcal{M}}$ which is an arbitrary generalized inverse of $\mathbf{\Gamma}_{\mathcal{M}} = \begin{pmatrix} \mathbf{V}_{\mathcal{M}} & \mathbf{X} \\ \mathbf{X}' & \mathbf{0} \end{pmatrix}$, expressed as (1.7). Notice that the rank of $\mathbf{\Gamma}_{\mathcal{M}}$ is

$$\text{rank}(\mathbf{\Gamma}_{\mathcal{M}}) = \text{rank}(\mathbf{X} : \mathbf{V}_{\mathcal{M}}) + \text{rank}(\mathbf{X}), \tag{5.1}$$

and therefore $\mathbf{\Gamma}_{\mathcal{M}}$ is invertible if and only if $\mathbf{X}$ has full column rank and $\text{rank}(\mathbf{X} : \mathbf{V}_{\mathcal{M}}) = n$.

The beauty and elegance in $\mathbf{C}_{\mathcal{M}}$ is that once it is calculated, then, as stated by Rao (1971, p. 378), we seem to have a Pandora's Box supplying all ingredients needed for obtain the BLUEs, their variances, and covariances, and constructing test criteria without any further computations except for a few matrix multiplications.

According to Wikipedia, Pandora's Box is an artifact in Greek mythology, taken from the myth of Pandora's creation in Hesiod's *Works and Days*. The "box" was actually a large jar given to Pandora which contained all the evils of the world. Today, however, the phrase "to open Pandora's Box" means to perform an action that may seem small or innocuous, but that turns out to have severe and far-reaching consequences. Certainly this latter interpretation fits appropriately into our considerations. A number of papers has been written on the problems presented in our paper but using different approach: our contribution lies in using the Rao's Pandora's technique to characterise the equality questions in two linear models.

[1] Alalouf, I. S. & Styan, G. P. H. (1979). Characterizations of estimability in the general linear model. *The Annals of Statistics*, 7, 194–200.

[2] Baksalary, J. K. & Kala, R. (1981). Linear transformations preserving best linear unbiased estimators in a general Gauss–Markoff model. *The Annals of Statistics*, 9, 913–916.

[3] Baksalary, J. K. & Mathew, T. (1986). Linear sufficiency and completeness in an incorrectly specified general Gauss–Markov model. *Sankhyā, Ser. A*, 48, 169–180.

[4] Baksalary, J. K., Rao, C. R. & Markiewicz, A. (1992). A study of the influence of the "natural restrictions" on estimation problems in the singular Gauss–Markov model. *Journal of Statistical Planning and Inference*, 31, 335–351.

[5] Baksalary, O. M. & Trenkler, G. (2009). A projector oriented approach to the best linear unbiased estimator. *Statistical Papers*, 50, 721–733.

[6] Baksalary, O. M. & Trenkler, G. (2011). Between OLSE and BLUE. *Australian & New Zealand Journal of Statistics*, 53, 289–303.

[7] Farebrother, R. W. (1979). Estimation with aggregated data. *Journal of Econometrics*, 10, 42–55.

[8] Gross, J. & Puntanen, S. (2000). Estimation under a general partitioned linear model. *Linear Algebra and its Applications*, 321, 131–144.

[9] Gross, J. & Trenkler, G. (1997). When do linear transforms of ordinary least squares and Gauss–Markov estimator coincide?. *Sankhyā, Ser. A*, 59, 175–178.

[10] Gross, J., Trenkler, G. & Werner, H. J. (2001). The equality of linear transformations of the ordinary least squares estimator and the best linear unbiased estimator. *Sankhyā, Ser. A*, 63, 118–127.

[11] Hall, F. J. & Meyer, C. D. (1975). Generalized inverses of the fundamental bordered matrix used in linear estimation. *Sankhyā, Ser. A*, 37, 428–438. [Corrigendum (1978), 40, p. 399.]

[12] Harville, D. A. (1997). *Matrix Algebra From a Statistician's Perspective*. Springer, New York.

[13] Haslett, S. J. (1996). Updating linear models with dependent errors to include additional data and/or parameters. *Linear Algebra and its Applications*, 237/238, 329–349.

[14] Haslett, S. J. & Puntanen, S. (2010). Effect of adding regressors on the equality of the BLUEs under two linear models. *Journal of Statistical Planning and Inference*, 140, 104–110.

[15] Hauke J., Markiewicz A. & Puntanen S. (2012) Comparing the BLUEs under two linear models. *Communications in Statistics: Theory and Methods*, 41, 2405–2418.

[16] Isotalo, J. & Puntanen, S. (2009). A note on the equality of the OLSE and the BLUE of the parametric functions in the general Gauss–Markov model. *Statistical Papers*, 50, 185–193.

[17] Isotalo, J., Puntanen, S. & Styan, G. P. H. (2007). Effect of adding regressors on the equality of the OLSE and BLUE. *International Journal of Statistical Sciences*, 6, 193–201.

[18] Lucke, B. (1991). On BLU-estimation with data of different periodicity. *Economics Letters*, 35, 173–177.

[19] Mathew, T. & Bhimasankaram, P. (1983). On the robustness of LRT in singular linear models. *Sankhyā, Ser. A*, 45, 301–312.

[20] Mitra, S. K. & Moore, B. J. (1973). Gauss–Markov estimation with an incorrect dispersion matrix. *Sankhyā, Ser. A*, 35, 139–152.

[21] Nurhonen, M. & Puntanen, S. (1992). Effect of deleting an observation on the equality of the OLSE and BLUE. *Linear Algebra and its Applications*, 176, 131–136.

[22] Puntanen, S. & Styan, G. P. H. (1990). Reply [to R. Christensen (1990), R. W. Farebrother (1990), and D. A. Harville (1990)]. (Letter to the Editor). *The American Statistician*, 44, 192–193.

[23] Puntanen, S., Styan, G. P. H. & Isotalo, J. (2011). *Matrix Tricks for Linear Statistical Models: Our Personal Top Twenty*. Springer, Heidelberg.

[24] Rao, C. R. (1967). Least squares theory using an estimated dispersion matrix and its application to measurement of signals. In: Le Cam, L. M., Neyman, J., eds. *Proc. Fifth Berkeley Symposium on Mathematical Statistics and Probability: Berkeley, California, 1965/1966*. Vol. 1. Univ. of California Press, Berkeley, pp. 355–372.

[25] Rao, C. R. (1971). Unified theory of linear estimation. *Sankhyā, Ser. A*, 33, 371–394. [Corrigendum (1972), 34, p. 194 and p. 477.]

[26] Rao, C. R. (1972a). A note on the IPM method in the unified theory of linear estimation. *Sankhyā, Ser. A*, 34, 285–288.

[27] Rao, C. R. (1972b). Some recent results in linear estimation. *Sankhyā, Ser. B*, 34, 369–378.

[28] Rao, C. R. (1973a). *Linear Statistical Inference and its Applications, 2nd Ed.* Wiley, New York.

[29] Rao, C. R. (1973b). Representations of best linear unbiased estimators in the Gauss–Markoff model with a singular dispersion matrix. *Journal of Multivariate Analysis*, 3, 276–292.

[30] Rao, C. R. & Mitra, S. K. (1971). *Generalized Inverse of Matrices and Its Applications.* Wiley, New York.

[31] Sengupta, D. & Jammalamadaka, S. R. (2003). *Linear Models: An Integrated Approach*. World Scientific, River Edge, NJ.

[32] Stahlecker, P. & Schmidt, K. (1987). On least squares estimation with a particular linear function of the dependent variable. *Economics Letters* , 23, 59–64.

[33] Tian, Y. & Puntanen, S. (2009). On the equivalence of estimations under a general linear model and its transformed models. *Linear Algebra and its Applications*, 430, 2622–2641.

[34] Tian, Y. & Zhang J. (2011). Some equalities for estimations of partial coefficients under a general linear regression model. *Statistical Papers*, 52, 911–920.

[35] Werner, H. J. (1987). C. R. Rao's IPM method: a geometric approach. *New Perspectives in Theoretical and Applied Statistics* (M. L. Puri, J. P. Vilaplana & W. Wertz, eds.). Wiley, New York, pp. 367–382.

[36] Werner, H. J. & Yapar, C. (1996). Some equalities for estimations of partial coefficients under a general linear regression model. *Linear Algebra and its Applications*, 237/238, 395–404.

[37] Zhang, B. (2007). The BLUE and MINQUE in Gauss–Markoff model with linear transformation of the observable variables. *Acta Mathematica Scientia, Series B*, 27, 203–210.

[38] Zhang, B., Liu, B. & Lu, C. (2004). A study of the equivalence of the BLUEs between a partitioned singular linear model and its reduced singular linear models. *Acta Mathematica Sinica, English Series, Vol. 20*, 3, 557–568.

[39] Zyskind, G. (1967). On canonical forms, non-negative covariance matrices and best and simple least squares linear estimators in linear models. *The Annals of Mathematical Statistics*, 38, 1092–1109.

THANK YOU VERY MUCH FOR YOUR ATTENTION!